

Holographic thermodynamics at finite baryon density: some exact results

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ABSTRACT: We use the AdS/CFT correspondence to study the thermodynamics of massive $\mathcal{N} = 2$ supersymmetric hypermultiplets coupled to $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills theory in the limits of large N_c and large 't Hooft coupling. In particular, we study the theory at finite baryon number density. At zero temperature, we present an exact expression for the hypermultiplets' leading-order contribution to the free energy, and in the supergravity description we clarify which D-brane configuration is appropriate for any given value of the chemical potential. We find a second-order phase transition when the chemical potential equals the mass. At finite temperature, we present an exact expression for the hypermultiplets' leading-order contribution to the free energy at zero mass.

KEYWORDS: Gauge-gravity correspondence, Brane Dynamics in Gauge Theories, AdS-CFT Correspondence, D-branes.

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1. Introduction and review

Understanding the phase diagram of Quantum Chromodynamics (QCD) is essential for understanding a variety of physical systems, such as the early universe and neutron stars. Much of the interesting thermodynamics of QCD, such as the deconfinement transition, occurs when traditional perturbative methods are unreliable. The lattice formulation of the theory circumvents this problem by direct calculation of the path integral. Implementing a baryon number chemical potential on the lattice is difficult, however. We are thus led to study the thermodynamics of model theories similar to QCD, but that offer more tractable strong-coupling analysis. In particular, the AdS/CFT correspondence, and more generally gauge-gravity duality, permits quantitative study of the thermodynamics of strongly-coupled non-Abelian gauge theories that share some features with QCD.

In this paper we will use the AdS/CFT correspondence to study the $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills (SYM) theory, in the 't Hooft limit of large- N_c , when the 't Hooft coupling is large, $\lambda \equiv g_{YM}^2 N_c \gg 1$. The $\mathcal{N} = 4$ SYM theory is very different from QCD at zero temperature, being a conformal field theory (CFT) and hence lacking confinement, a mass gap, and chiral symmetry breaking. At finite temperature, however, the $\mathcal{N} = 4$ SYM theory resembles the high-temperature phase of QCD, in particular by exhibiting screening of color charges. The $\mathcal{N} = 4$ SYM theory at finite temperature has no phase transitions, however. Being a CFT, the theory has no intrinsic scale to set a transition temperature, so no transitions can occur.

The AdS/CFT correspondence is the conjecture that the $\mathcal{N} = 4$ SYM theory, in the limits described above, is equivalent to type IIB supergravity formulated on the background spacetime $AdS_5 \times S^5$, where AdS_5 is five-dimensional anti-de Sitter space and S^5 is a five-sphere [1–3]. This spacetime arises as the near-horizon geometry of $N_c \rightarrow \infty$ coincident D3-branes in type IIB string theory. Thermal equilibrium of the $\mathcal{N} = 4$ SYM theory corresponds to non-extremal D3-branes, whose near-horizon geometry is five-dimensional AdS-Schwarzschild times S^5 [4, 5]. The temperature T of the $\mathcal{N} = 4$ SYM theory is identified with the Hawking temperature of the AdS-Schwarzschild black hole.

The $\mathcal{N} = 4$ SYM theory, unlike QCD, only describes fields in the adjoint representation of the gauge group. We will introduce fields in the fundamental representation in the form of $\mathcal{N} = 2$ supersymmetric hypermultiplet “flavor” fields. We will introduce a finite number N_f of them, so that $N_f \ll N_c$ in the ’t Hooft limit, and work to leading order in N_f/N_c . To this order, the beta function of the theory is zero and hence the theory remains (approximately) conformal. We may introduce an $\mathcal{N} = 2$ supersymmetry-preserving mass m for the hypermultiplets, however, which of course explicitly breaks the conformal invariance.

The N_f hypermultiplets possess a $U(N_f)$ global symmetry analogous to the vector flavor symmetry of QCD. As in QCD, we identify the $U(1)_B$ subgroup of $U(N_f)$ as baryon number. We will introduce a finite $U(1)_B$ density $\langle J^t \rangle$, where J^μ is the $U(1)_B$ current, or equivalently we work with a finite chemical potential μ .

The flavor fields appear in the supergravity description as N_f D7-branes [6]. In the $N_f \ll N_c$ limit, the contribution that these D7-branes make to the stress-energy tensor is dwarfed by the contribution of the N_c D3-branes. To leading order in N_f/N_c , then, we may neglect the back-reaction of the D7-branes on the geometry: they are probes. The D7-brane action is then the Dirac-Born-Infeld (DBI) action. The hypermultiplet mass m appears in the supergravity description as the asymptotic separation of the D3-branes and D7-branes in a mutually orthogonal direction. The $U(N_f)$ flavor symmetry appears in the supergravity description as the $U(N_f)$ worldvolume gauge invariance of the N_f D7-branes. The $U(1)_B$ current J^μ is dual to the $U(1)$ worldvolume gauge field A_μ . To obtain a finite baryon number density $\langle J^t \rangle$ in the SYM theory, we must introduce a nonzero gauge field time component, $A_t(r)$, where r is the radial coordinate of AdS.

The SYM theory has three dimensionful parameters: the temperature, T , the hypermultiplet mass m , and the baryon number density, $\langle J^t \rangle$. We will use the D7-brane description of the flavor fields to derive exact expressions for their free energy in two limits: $T = 0$ and $m = 0$. To place our results in context, we will now review the known thermodynamics of the flavor fields in the SYM theory, as deduced from supergravity calculations.

Consider first zero temperature, zero mass, and zero density in the SYM theory. The D3-branes and D7-branes then have zero asymptotic separation. The D7-branes are extended along $AdS_5 \times S^3$ inside of $AdS_5 \times S^5$, where the worldvolume S^3 is the maximum-volume equatorial $S^3 \subset S^5$ [6]. We first turn on m , which means finite asymptotic separation of the D3-branes and D7-branes. The D7-brane then has a non-trivial embedding in the near-horizon geometry. The D7-brane wraps the equatorial $S^3 \subset S^5$ at the boundary of AdS_5 . Moving into the bulk of AdS_5 , the position of the S^3 can change, that is, the D7-brane “slips” on the S^5 . The volume of the S^3 shrinks as we move into AdS_5 and

indeed may ultimately vanish at some radial position r' , as allowed by topology (the S^3 is a trivial cycle). The D7-brane then does not extend past r' : it appears to “end” at r' [6]. The worldvolume scalar that describes the embedding is dual holographically to an operator \mathcal{O}_m given by taking $\frac{\partial}{\partial m}$ of the SYM Lagrangian. In particular, \mathcal{O}_m includes the mass operator of the fermions in the hypermultiplet as well as m times the mass operator of the scalars, as well as couplings to adjoint scalars. The exact operator is written in ref. [7]. For our purposes, thinking of \mathcal{O}_m as the mass operator will be sufficient.

Now turn on the temperature T . The spacetime is now AdS-Schwarzschild, and a horizon is present at some radial position r_H . Two classes of embedding are now possible. The first are the finite-temperature analogues of the embeddings described above: D7-branes that end outside the horizon, i.e. $r' > r_H$. These are called “Minkowski” embeddings. The second class are D7-branes in which the $S^3 \subset S^5$ shrinks as we move away from the boundary, but never collapses to zero volume. The D7-brane then extends to, and intersects, the AdS-Schwarzschild horizon. These are called “black hole” embeddings. A topology-changing phase transition exists between these two types of embeddings, which appears in the SYM theory as a first-order phase transition associated with the flavor fields as we change m/T [8–15].

Now turn on the baryon number density $\langle J^t \rangle$. The U(1) gauge field on the D7-brane worldvolume now has nonzero $A_t(r)$. The asymptotic value $A_t(\infty)$ gives the value of μ . In the canonical ensemble, the topology-changing transition observed at zero density persists for small density and then ends in a critical point [7]. A region of instability exists, with the line of first-order transitions as a boundary [7].

A claim was made in ref. [7] that when F_{tr} is nonzero only black hole embeddings are physically allowed. Minkowski embeddings were dismissed as unphysical because, if the D7-brane ends outside the horizon, the radial electric field lines F_{tr} have no place to end. If a source, a density $\langle J^t \rangle$ of fundamental strings stretching from the D7-brane to the horizon, is introduced to accommodate the field lines, the tension of the strings will overcome the tension of the D7-brane, drawing the D7-brane into the horizon and producing a black hole embedding. Minkowski embeddings were thus excluded from the analysis of ref. [7].

In refs. [16, 17], however, an argument was made that Minkowski embeddings were physical and should be included. One of the main objections to the claim of ref. [7] was that the black hole configurations cannot realize chemical potentials below a critical value, which at zero temperature we will show is $\mu = m$. What we find is that this “incompleteness problem” is completely resolved by including Minkowski branes with a *constant* $A_t(r)$, that is, with non-vanishing chemical potential but zero density.¹ The authors of refs. [16, 17] argue that A_t is not a gauge-invariant parameter, but in the thermodynamic setting where spacetime has a compact Euclidean time direction, the Wilson line of A_t is gauge invariant and hence physical. The constant part in $A_t(r)$ is the chemical potential according to the standard rules of AdS/CFT, since it represents the leading near-boundary behavior. For any configuration that satisfies the boundary condition that $A_t(r)$ vanish in the infrared (deep in AdS), this definition becomes equivalent to the description $\mu = \int F_{rt}$ advertised in

¹While this paper was in preparation ref. [18] appeared, where the same conclusion was reached.

refs. [16, 17]. The latter definition misses the important case that $A_t(r)$ is simply constant, however, which is always a solution in the absence of sources.

We will find analytic solutions for the zero-temperature limit of black hole embeddings, and will argue that these are thermodynamically favored when $\mu > m$ and $\langle J^t \rangle > 0$. This is in complete agreement with the claims of ref. [7], where only $\langle J^t \rangle > 0$ was considered. When $\mu \leq m$, Minkowski embeddings with constant $A_t(r)$, and hence $\langle J^t \rangle = 0$, are thermodynamically favored. The transition between the two is second order. Indeed, when $\mu > m$ we see, as expected, a free energy whose form is consistent with spontaneous symmetry breaking for a scalar field with a wine-bottle potential. When $\mu = m$ the minimum of the scalar's potential returns to the origin. Note in particular that we will not see a Fermi surface: the system prefers to condense scalars.

We can also generalize our zero-temperature analysis to include nonzero currents $\langle J^i \rangle$, for spatial index i . At zero temperature, these currents will not dissipate and can be introduced as a constant background. We can generalize further, from a D7-brane probe to a D5-brane probe [19, 20]. Indeed, our analysis should generalize to many probe D-brane systems.

At finite temperature, we are also able to provide an analytic expression for the hypermultiplets' contribution to the free energy at leading order in N_f/N_c in the limit $m = 0$. The result exhibits no interesting phase structure, in accord with the phase diagram of ref. [7].

Another holographic model of QCD at finite baryon density, the D4/D8 system, and specifically the Sakai-Sugimoto model [21], has received a great deal of attention recently [22–29]. In this case, the supergravity background is provided by D4-branes and the flavor branes are D8-branes and anti-D8-branes. As in ref. [30], in the course of our analysis we will find a D7-brane configuration similar to the D8-brane in the Sakai-Sugimoto model, but, also as in ref. [30], we will find that these configurations are always thermodynamically disfavored. Indeed, these solutions should not be compared to any of our other solutions, as they are non-supersymmetric even at zero density and describe a SYM Lagrangian distinctly different from that of the SYM theory described above.

Most of the work on the Sakai-Sugimoto model at finite density is based on analogues of Minkowski embeddings, with strings attached (or alternatively a wrapped D4-brane providing the required source). Our results, as well as the earlier work of ref. [7], show that in the analytically tractable D3/D7 system, such configurations are not solutions of the full source-plus-brane system, and do not even provide a good qualitative picture of the phase structure. This at least suggests that any analysis based on approximating a full solution to the non-abelian DBI action relevant in the Sakai-Sugimoto model with a source-plus-brane configuration has to be approached with some caution. Presumably a solution like the smooth black hole embedding for the D7-brane discussed here exists in that case as well, but it most likely will involve non-abelian configurations of scalars and gauge fields as well as the tachyon.

This paper is organized as follows. In section 2 we present our zero-temperature analysis. In section 3 we present our zero mass analysis. We conclude in section 4. We collect various necessary integrals in the appendix.

2. The limit of zero temperature

2.1 Analytic brane embeddings

In type IIB string theory, we consider a system of N_c D3-branes and N_f D7-branes aligned in flat ten-dimensional space as

$$\begin{array}{cccccccccc}
 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\
 \text{D3} & \times & \times & \times & \times & & & & & & \\
 \text{D7} & \times & \times & \times & \times & \times & \times & \times & \times & &
 \end{array} \tag{2.1}$$

If we separate the D3-branes and D7-branes in a mutually orthogonal direction, in the x_8 - x_9 plane, an open string stretched between them will have a mass equal to the separation times the string tension. This mass appears as the hypermultiplet mass m in the D3-brane worldvolume theory. We take the usual AdS/CFT limit, $N_c \rightarrow \infty$, $g_s \rightarrow 0$ with $g_s N_c$ fixed and $g_s N_c \gg 1$ [1]. We obtain the near-horizon geometry of D3-branes, $AdS_5 \times S^5$, with the $N_f \ll N_c$ D7-branes extended along $AdS_5 \times S^3$.

The DBI action describing the embedding of any probe D-brane is non-linear. Even when the symmetries are strong enough to restrict all the embedding functions to depend on a single variable only, the resulting ordinary differential equation of motion can often only be solved numerically. Sometimes symmetries provide integrals of motion that permit an exact solution of the system. For example, in the Sakai-Sugimoto model [21] translation invariance along the direction x transverse to the D8-branes ensures that the D8-brane action only depends on derivatives of x and not on x itself. We will show that the same is true for the D3/D7 system at zero temperature even in the presence of certain worldvolume gauge fields. From the SYM perspective, this is the statement that we may determine the free energy exactly for non-vanishing currents J^μ , and in particular for finite density $\langle J^t \rangle$.

In the original analysis of ref. [6], the embedding of the D7-brane in $AdS_5 \times S^5$ was characterized by a slipping mode $\theta(r_6)$. That is, the $AdS_5 \times S^5$ metric was written as

$$ds^2 = \frac{r_6^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r_6^2} dr_6^2 + (d\theta^2 + \sin^2 \theta ds_{S^1}^2 + \cos^2 \theta ds_{S^3}^2) \tag{2.2}$$

where $\mu, \nu = 0, 1, 2, 3$ denote the four worldvolume directions of the D3-brane, $\eta_{\mu\nu}$ is the (3+1)-dimensional Minkowski metric, R is the curvature radius,² $ds_{S^1}^2$ and $ds_{S^3}^2$ are the metrics of unit 1- and 3-spheres, respectively, and r_6 is the radial direction in the full space transverse to the D3-branes. In terms of the ten-dimensional space in which the N_c -D3 branes are embedded, $r_6^2 = \sum_a x_a^2$, where a runs from 4 to 9. The action of a D7-brane wrapping $AdS_5 \times S^3$ is proportional to the volume of the S^3 , which goes as $\cos^3 \theta$, so we will find a non-trivial potential for the embedding scalar $\theta(r_6)$ in this case, and no simple integral of motion is apparent. However, if we write the background metric as

$$ds^2 = Z(r_6) \eta_{\mu\nu} dx^\mu dx^\nu + Z^{-1}(r_6) (dr^2 + r^2 ds_{S^3}^2 + dy^2 + dz^2), \tag{2.3}$$

²Starting now, we work in units with $R \equiv 1$. In particular, we convert between supergravity and SYM quantities with $\alpha' = \lambda^{-1/2}$.

a significant simplification occurs. In this case, $y(r)$ and $z(r)$, which stand for x_8 and x_9 , are the scalars describing the embedding, and without loss of generality we can use the U(1) symmetry that rotates them into one another to set $z = 0$. r is a radial coordinate in four of the six transverse directions, so we have $r_6^2 = r^2 + y^2 + z^2$. All the y -dependence of the action appears via r_6 in the warp factors $Z(r_6)$. For the embeddings we are interested in, with a D7-brane wrapping the four x_μ directions and four of the six transverse directions, all the warp factors will drop out from the action, and we are effectively solving for D7-brane embeddings in flat space. The same simplification occurs in all supersymmetric probe brane embeddings where background and probe branes have four relative ND directions. The action now only depends on derivatives of y , and the resulting constant of motion is sufficient to integrate analytically the equations of motion. This cancellation of warp factors is spoiled at finite temperature or in the presence of electric or magnetic background fields in the SYM theory. This coordinate system was originally used in ref. [31] to find the exact meson spectrum of the SYM theory from fluctuations of the D7-brane. More recently, it was used in ref. [30] to find new non-supersymmetric D7/anti-D7-brane embeddings. To translate back to the original $\theta(r)$ embedding scalar, we can always combine the definitions $y = r_6 \sin \theta$ and (at $z = 0$) $r_6^2 = r^2 + y^2$ to obtain $\tan \theta = \frac{y(r)}{r}$.

The action for the D7-branes is

$$S_{D7} = -N_f T_{D7} \int d^8 \xi \sqrt{-\det(g_{ab} + (2\pi\alpha') F_{ab})} \tag{2.4}$$

where T_{D7} is the D7-brane tension, ξ_a are worldvolume coordinates, g_{ab} is the induced worldvolume metric and F_{ab} is the worldvolume U(1) gauge field. Introducing the field strength³ $F_{rt} = A'_t(r)$ and the embedding scalar $y(r)$, the zero-temperature D7-brane Lagrangian is

$$L = -\mathcal{N} r^3 \sqrt{1 + y'^2 - (2\pi\alpha')^2 A_t'^2}. \tag{2.5}$$

where $\mathcal{N} = N_f T_{D7} V_3$ with $V_3 = 2\pi^2$ the volume of a unit S^3 . In terms of SYM theory quantities,

$$\mathcal{N} = \frac{\lambda}{(2\pi)^4} N_f N_c \tag{2.6}$$

We will divide both sides of eq. (2.4) by the volume of $\mathbb{R}^{3,1}$ and henceforth work with action (and in a moment, free energy) densities, so that $S_{D7} = \int dr L$.

Thanks to the warp factor cancellation, only derivatives of $y(r)$ and $A_t(r)$ appear in the Lagrangian, and we find two conserved charges

$$\frac{\delta L}{\delta y'} = -\mathcal{N} r^3 \frac{y'}{\sqrt{1 + y'^2 - (2\pi\alpha')^2 A_t'^2}} \equiv -c, \quad \frac{\delta L}{\delta A_t'} = \mathcal{N} r^3 \frac{(2\pi\alpha')^2 A_t'}{\sqrt{1 + y'^2 - (2\pi\alpha')^2 A_t'^2}} \equiv d. \tag{2.7}$$

The ratio implies

$$A_t'^2 = \frac{d^2}{(2\pi\alpha')^4 c^2} y'^2. \tag{2.8}$$

³Throughout this paper, we work in a gauge with $A_r = 0$.

We can algebraically solve for $y'(r)$ and $A'_t(r)$ in terms of the integration constants c and d ,

$$y' = \frac{c}{\sqrt{\mathcal{N}^2 r^6 + \frac{d^2}{(2\pi\alpha')^2} - c^2}}, \quad A'_t = \frac{d/(2\pi\alpha')^2}{\sqrt{\mathcal{N}^2 r^6 + \frac{d^2}{(2\pi\alpha')^2} - c^2}}. \quad (2.9)$$

These can be integrated using incomplete Beta functions (whose properties we review in the appendix). The result depends on the sign of $\frac{d^2}{(2\pi\alpha')^2} - c^2$. When $c = d = 0$, we obtain the solution with $y'(r) = 0$ and $A'_t(r) = 0$, so $y(r)$ and $A_t(r)$ are constants. If $A_t(r) = 0$, we recover the original embedding of ref. [6] for a single straight D7-brane. This embedding is depicted in figure 1a. Solutions with $\frac{d^2}{(2\pi\alpha')^2} - c^2$ positive and negative are also depicted in figure 1. We address each of these in detail in the following sections.

The action evaluated on these solutions is

$$S_{D7} = -\mathcal{N} \int_0^\Lambda dr r^3 \sqrt{\frac{\mathcal{N}^2 r^6}{\mathcal{N}^2 r^6 + \frac{d^2}{(2\pi\alpha')^2} - c^2}} \quad (2.10)$$

The lower endpoint of integration will depend on the sign of $\frac{d^2}{(2\pi\alpha')^2} - c^2$. The integral diverges if we integrate to $r = \infty$, so we regulate the integral with a cutoff at $r = \Lambda$. The divergence is present for $d = c = 0$, i.e. for straight D7-branes with $A_t(r) = 0$,

$$S_0 = -\mathcal{N} \int_0^\Lambda dr r^3 = -\frac{1}{4} \mathcal{N} \Lambda^4 \quad (2.11)$$

In what follows we will obtain the renormalized on-shell action, S_{ren} , by subtracting this divergence,

$$S_{\text{ren}} = \lim_{\Lambda \rightarrow \infty} (S_{D7} - S_0) \quad (2.12)$$

In the SYM theory, the thermodynamic potential Ω of the grand canonical ensemble is given by $\Omega = -S_{\text{ren}}$. We Legendre transform to obtain the free energy density in the canonical ensemble, $F = \Omega + \mu d$. Recall that at zero temperature, free energy and energy are the same, so at zero temperature F is also the energy density. The conserved charges c and d determine $\langle \mathcal{O}_m \rangle$ and $\langle J^t \rangle$ as follows,

$$\langle \mathcal{O}_m \rangle = \frac{\delta \Omega}{\delta m} = -(2\pi\alpha') \frac{\delta S_{D7}}{\delta y(\infty)}, \quad \langle J^t \rangle = -\frac{\delta \Omega}{\delta \mu} = \frac{\delta S_{D7}}{\delta A_t(\infty)} \quad (2.13)$$

where in each case when we vary one field we hold the other fixed. We then have

$$\delta S_{D7} = \int dr \left(\frac{\delta L}{\delta A'_t(r)} \partial_r \delta A_t(r) + \frac{\delta L}{\delta y'(r)} \partial_r \delta y(r) \right) = d \delta A_t(\infty) - c \delta y(\infty) \quad (2.14)$$

where we demand that $\delta A_t(r)$ and $\delta y(r)$ are always zero at the lower endpoint of the r integration. If we vary $A_t(r)$ while holding $y(r)$ fixed ($\delta y(r) = 0$), we find $\langle J^t \rangle = d$, and similarly we find $\langle \mathcal{O}_m \rangle = (2\pi\alpha')c$. Starting now, we will stick to the notation c and d and refer to these as the condensate and density.

Using $\Omega = -S_{\text{ren}}$, we can determine whether solutions with nonzero d and c are thermodynamically favored relative to the single straight D7-brane with zero gauge field. $S_{\text{ren}} > 0$ means the configuration is *favored* and $S_{\text{ren}} < 0$ means the configuration is *disfavored*.

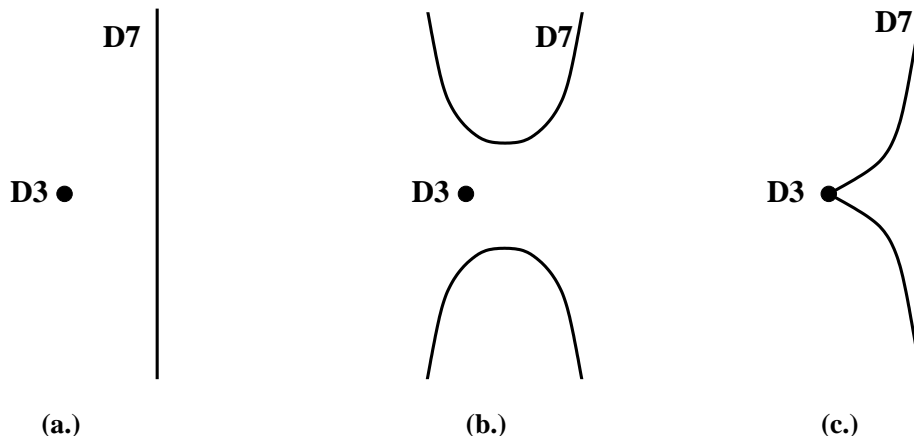


Figure 1: Cartoons of the allowed D7-brane embeddings at zero temperature. The vertical axis is one of the directions transverse to the D3-branes but parallel to the D7-branes. The horizontal axis is an overall transverse direction. (a.) A Minkowski embedding corresponding to $y(r)$ being constant. (b.) A $\frac{d^2}{(2\pi\alpha')^2} - c^2 < 0$ solution with turnaround describing a brane/anti-brane embedding as in ref. [30]. (c.) The zero-temperature $\frac{d^2}{(2\pi\alpha')^2} - c^2 > 0$ horizon-crossing “black hole” embedding.

2.2 Brane/Anti-brane embeddings

We begin with $\frac{d^2}{(2\pi\alpha')^2} - c^2 < 0$. In particular, we will begin with $d = 0$ solutions, as analyzed recently in ref. [30]. For such $d = 0$ embeddings, $y'(r)$ diverges as $\frac{1}{\sqrt{r-r_0}}$ at a critical radius $r_0 = \mathcal{N}^{-1/3} c^{1/3}$. This behavior indicates that the D7-brane has a turn-around point where the D7-brane smoothly matches onto a second branch. This solution thus describes a D7/anti-D7-brane pair connected by a smooth neck, as depicted schematically in figure 1b.

In the asymptotic region, we may specify the distances of the D7-brane and the anti-D7-brane to the D3-branes separately. In other words, we may associate two masses to such a configuration. We denote these as m and \bar{m} . A D7-brane solution exists for any m and $\langle \mathcal{O}_m \rangle$. The mass (or asymptotic separation) of the anti-D7-brane is then fixed in terms of these two input parameters. From the field theory point of view, we would like to interpret the two masses m and \bar{m} as input parameters, so that the above construction dynamically determines the value of the condensate $\langle \mathcal{O}_m \rangle$. From the supergravity picture, we can see that the condensate only depends on the difference of the masses, $m - \bar{m}$. This is another reflection of the fact that the embedding equation lost all knowledge of the warp factors, and effectively we are solving for flat-space embeddings. Only the relative position of the D7-branes and anti-D7-branes matters, not their distance from the origin. As noted in ref. [30], however, all the connected D7/anti-D7-brane solutions have higher energy than the straight D7/anti-D7-brane pair, and hence will be meta-stable, at best. The true ground state of the system has vanishing $\langle \mathcal{O}_m \rangle$.

Now we introduce nonzero d . To connect the two halves of the D7-brane smoothly we need the gauge field and its derivative to be continuous at the neck. In particular, the field strength on the two branches should be equal in magnitude but opposite in direction. The

baryon number density is then equal and opposite at the two ends, so this configuration does not describe any net baryon number. The Lagrangian has a global $U(1) \times U(1)$ symmetry, and, at least in the grand canonical ensemble, we can imagine turning on different chemical potentials for the two $U(1)$'s at the level of the Lagrangian, even though the symmetry is spontaneously broken to the diagonal $U(1)_B$ in the state represented by the connected D7-brane configuration. If we wanted to have a brane/anti-brane system with finite baryon density we would have to include an explicit source at the neck. Such additional sources will back-react and alter the configuration, so we will not discuss this option further.

In any case, these brane/anti-brane systems correspond to a different Lagrangian than that for systems with a single D7-brane reaching the asymptotic region. The dual field theory for the brane/anti-brane is $\mathcal{N} = 4$ SYM coupled to two hypermultiplets that preserve opposite $\mathcal{N} = 2$ supersymmetries, so that the full theory is non-supersymmetric. When comparing all possible bulk configurations that correspond to hypermultiplets preserving $\mathcal{N} = 2$ supersymmetry, the brane/anti-brane configurations never contribute. We will therefore only compare these solutions to the straight D7-brane and anti-D7-brane.

For nonzero d , keeping in mind that for the brane/anti-brane system this does not correspond to a net baryon density, the D7-brane reaches the turn-around point

$$r_0 = \mathcal{N}^{-1/3} \left(c^2 - \frac{d^2}{(2\pi\alpha')^2} \right)^{1/6} \tag{2.15}$$

The integrals for $y(r)$, $A_t(r)$ and the regulated on-shell action are performed in the appendix. We find

$$y(r) = c \frac{1}{6} \mathcal{N}^{-1/3} \left(c^2 - \frac{d^2}{(2\pi\alpha')^2} \right)^{-1/3} \left(B \left(\frac{1}{3}, \frac{1}{2} \right) - B \left(\frac{r_0^6}{r^6}; \frac{1}{3}, \frac{1}{2} \right) \right) \tag{2.16}$$

and $A_t(r)$ is simply $\frac{d}{(2\pi\alpha')^2} \frac{1}{c}$ times $y(r)$, as shown in eq. (2.8). Notice $y(r_0) = A_t(r_0) = 0$. The mass and chemical potential are given by the asymptotic values at $r \rightarrow \infty$, with $\lim_{r \rightarrow \infty} B \left(\frac{r_0^6}{r^6}; \frac{1}{3}, \frac{1}{2} \right) = 0$. The renormalized on-shell action for the D7-brane is

$$S_{\text{ren}} = -\frac{1}{6} \mathcal{N}^{-1/3} \left(c^2 - \frac{d^2}{(2\pi\alpha')^2} \right)^{2/3} B \left(-\frac{2}{3}, \frac{1}{2} \right) \tag{2.17}$$

The action for the anti-D7-brane is identical, so the total action is twice S_{ren} . Notice that $S_{\text{ren}} < 0$, so these connected brane/anti-brane configurations always have a higher free energy than the straight brane and anti-brane with constant $A_t(r)$, and hence are thermodynamically disfavored, just as in the $d = 0$ case.

At this point we may also consider $c = \frac{d}{2\pi\alpha'} \neq 0$ so that $\frac{d^2}{(2\pi\alpha')^2} - c^2 = 0$. This configuration shares a similar feature with the brane/anti-brane configurations: it does not satisfy the required UV boundary conditions for a theory with a single D7-brane added. When $\frac{d^2}{(2\pi\alpha')^2} - c^2 = 0$, we have $y'(r) \sim \frac{1}{r^3}$, so the D7-brane misses the D3-branes at the origin and returns to the asymptotic region at $r \rightarrow \infty$. We will therefore not study the $c = \frac{d}{2\pi\alpha'}$ case further.

2.3 Finite-density black hole embeddings

We now consider $\frac{d^2}{(2\pi\alpha')^2} - c^2 > 0$. In this section we will show that these are the zero-temperature limit of black hole embeddings. First, notice that these solutions have no turn-around point. They describe D7-branes alone.

For a $\frac{d^2}{(2\pi\alpha')^2} - c^2 > 0$ solution, suppose we take $c = 0$. The solution with constant $y(r)$ is still allowed. Geometrically, this is the zero-temperature Minkowski embedding, but now with nonzero $A'_t(r)$. As mentioned before, the constant charge density along the D7-brane requires a source at $r = 0$, so this embedding is not physical unless we add extra strings connecting the D3-branes and D7-branes, or some other source. We will return to these $c = 0$ embeddings in the next subsection.

We can avoid this issue, however, by demanding that the D7-brane touch the horizon at $r = 0$. Since the horizon is located at $0 = r_6^2 = r^2 + y^2$, this immediately implies a boundary condition on y

$$y(r = 0) = 0. \tag{2.18}$$

The embedding satisfying this boundary condition is the zero temperature analogue of black hole embeddings. We will stick to this name, even though at zero temperature “horizon crossing” would be more appropriate. Using the change of variables $\tan \theta = y/r$, we see that the slope of $y(r)$ at $r = 0$ sets the value of θ at the horizon. This black hole embedding is displayed schematically in figure 1c.

The integral for $y'(r)$ in eq. (2.9) is done in the appendix, with the result

$$y(r) = c \frac{1}{6} \mathcal{N}^{-1/3} \left(\frac{d^2}{(2\pi\alpha')^2} - c^2 \right)^{-1/3} B \left(\frac{\mathcal{N}^2 r^6}{\mathcal{N}^2 r^6 + \frac{d^2}{(2\pi\alpha')^2} - c^2}; \frac{1}{6}, \frac{1}{3} \right) \tag{2.19}$$

and again, $A_t(r)$ is $\frac{d}{(2\pi\alpha')^2} \frac{1}{c}$ times $y(r)$. Notice that $y(0) = A_t(0) = 0$. Identifying m and μ from the asymptotic values $y(\infty)$ and $A_t(\infty)$, where as $r \rightarrow \infty$ the incomplete Beta function becomes $B\left(\frac{1}{6}, \frac{1}{3}\right)$, we find

$$\begin{aligned} c &= \gamma \mathcal{N} (2\pi\alpha')^3 (\mu^2 - m^2) m \\ \frac{d}{2\pi\alpha'} &= \gamma \mathcal{N} (2\pi\alpha')^3 (\mu^2 - m^2) \mu \end{aligned} \tag{2.20}$$

where we have defined the constant

$$\gamma \equiv \left(\frac{1}{6} B \left(\frac{1}{6}, \frac{1}{3} \right) \right)^{-3} \approx 0.363. \tag{2.21}$$

Note that since these black hole embeddings were valid only when $\frac{d^2}{(2\pi\alpha')^2} - c^2 > 0$, we can immediately see that they only exist for $\mu > m$. In a theory with a mass gap, chemical potentials of magnitude less than the mass gap are basically trivial: they do not lead to any finite density. In the canonical ensemble they never appear, since zero density corresponds to $\mu = m$. In the grand canonical ensemble we should of course be able to dial μ all the way down to zero, but the physics for $\mu < m$ is still expected to be trivial in this range, with $d = 0$ and the free energy being independent of μ . These expectations will be verified

when we study Minkowski embeddings in the next subsection. In this sense, the black hole embeddings cover the whole range of physically interesting values of μ .

The functional form of the condensate $c(\mu, m)$ is exactly that of a charged boson in the presence of a chemical potential. The chemical potential leads to a negative mass squared for the boson. Once this is larger than the positive mass squared in the Lagrangian, and if the scalar has a quartic coupling, the scalar will condense, its expectation value settling to the new minimum. Eqs. (2.20) can thus be viewed as determining the effective quartic potential at strong coupling. Notice that this also confirms the field theory expectation that at finite density the ground state is described by scalar condensates, not by a Fermi surface.

The renormalized on-shell action in this case is

$$S_{\text{ren}} = \frac{1}{4} \gamma^{-1/3} \mathcal{N}^{-1/3} \left(\frac{d^2}{(2\pi\alpha')^2} - c^2 \right)^{2/3} = \frac{1}{4} \gamma \mathcal{N} (2\pi\alpha')^4 (\mu^2 - m^2)^2 \quad (2.22)$$

Notice that $S_{\text{ren}} > 0$, so these embeddings are thermodynamically favored relative to the straight D7-brane with no gauge field. In the next subsection we will compare to the straight D7-brane with a nonzero gauge field. Notice that we observe no instability in Ω . Stability requires $\frac{\partial d}{\partial \mu} \geq 0$, which is clearly satisfied in eq. (2.20). This is consistent with the numerical results of ref. [7] in the zero-temperature limit.

The thermodynamic potential in the canonical ensemble is

$$F = \Omega + \mu d = \frac{1}{4} \mathcal{N}^{-1/3} \gamma^{-1/3} \left(\frac{d^2}{(2\pi\alpha')^2} - c^2 \right)^{-1/3} \left(3 \frac{d^2}{(2\pi\alpha')^2} + c^2 \right) \quad (2.23)$$

so that $F > 0$. We will see in the next subsection that the black hole embedding still turns out to be the configuration with lowest F , however.

2.4 Finite-density Minkowski embeddings

Let us return to $c = 0$ and constant $y(r)$. These are solutions everywhere away from $r = 0$. These are straight D7-branes, which do not obey the boundary condition $y(0) = 0$. In other words, when $r = 0$ the D7-branes are still a distance y away from the D3-branes. The mass m is given by $m = \frac{y}{2\pi\alpha'}$.

The integral for $A_t(r)$ will be unchanged, however. In particular, we still integrate from $r = 0$ to $r = \infty$, and $A_t(0) = 0$. From eq. (2.20) we read off the relation between μ and d ,

$$d = \gamma \mathcal{N} (2\pi\alpha')^4 \mu^3, \quad (2.24)$$

The integral for the on-shell action is also unchanged, so we simply set $c = 0$ in the first equality of eq. (2.22),

$$S_{\text{ren}} = \frac{1}{4} \gamma^{-1/3} \mathcal{N}^{-1/3} (2\pi\alpha')^{-4/3} d^{4/3} = \frac{1}{4} \gamma \mathcal{N} (2\pi\alpha')^4 \mu^4 \quad (2.25)$$

and hence $\Omega = -S_{\text{ren}} \sim -\mu^4$. Naively this Ω , for any nonzero d (or $\mu > m$), is smaller (more negative) than the one for the black hole embedding, eq. (2.22). Notice, however,

that unlike the nice interpretation we found for the relations among c , d , μ and m for the black hole embeddings, this result for Ω , a pure quartic in μ , is completely independent of the mass and therefore appears to be unphysical. For example, in the decoupling limit $m \rightarrow \infty$, Ω should go to zero.

The problem of course is that the Minkowski embedding, as it stands, is not consistent: we completely ignored the contribution to Ω due to whatever object sources the D7-brane gauge field. The simplest source is a finite density d of fundamental strings stretching from the D3-branes to the D7-brane. These D7-branes with strings attached are not solutions to the full equations of motion following from the combined DBI and Nambu-Goto actions, however. Naively, we might think that the string back-reaction on the D7-brane can be neglected in the large- N_c and large- λ limit. For a *single* string this is certainly true. The prefactor of the Nambu-Goto action scales as $\sqrt{\lambda}$, while from eq. (2.6) we see that the prefactor \mathcal{N} of the DBI action scales as $N_f N_c \lambda$. We want a finite density d of strings, however. If we keep all geometric distances of order one in units of the AdS curvature radius, the mass will scale as $\sqrt{\lambda}$, so we want the chemical potential also to scale as $\sqrt{\lambda}$. From eq. (2.20), we can see that the density d will then scale as $N_f N_c \sqrt{\lambda}$. With this, d times the Nambu-Goto action has the same $N_f N_c \lambda$ scaling as the DBI action, and so the back-reaction is order one.

To include the back-reaction one has to re-solve the equations of motion including the extra source term. We already found the most general solution to the equations of motion with sources localized at $r = 0$, so the back-reacted solution including any such sources must be within this class. The only well-behaved solution in this class which asymptotically becomes a single brane is the black hole embedding, so it must be the back-reacted solution.

To show that this is physically reasonable we want to show that the brane-plus-strings configuration, which could serve as a consistent initial data for a full time-dependent physical solution, has higher energy than the black hole embedding. We add a term dm to the action representing the finite density d of strings with length $y = (2\pi\alpha')m$. The full free energy including the D7-branes and strings in this case is

$$\frac{\Omega}{\frac{1}{4}\gamma\mathcal{N}(2\pi\alpha')^4} = -\mu^4 + 4\mu^3 m = \mu^3(4m - \mu) \tag{2.26}$$

which is positive at $\mu = m$ and is disfavored relative to the black hole embedding at all values of $\mu > m$. We can also allow some mixture, where the D7-brane embedding satisfies the boundary condition $y(r = 0) = y_0$ for some $m > \frac{y_0}{2\pi\alpha'} > 0$, with a density d of strings extending from $y = 0$ to $y = y_0$. In figure 1, we imagine sliding the black hole embedding to the right a distance y_0 , such that the total asymptotic separation remains y , and then introducing the strings. The free energy is then

$$\frac{\Omega}{\frac{1}{4}\gamma\mathcal{N}(2\pi\alpha')^4} = -\left(\mu^2 - \left(m - \frac{y_0}{2\pi\alpha'}\right)^2\right)^2 + 4\mu\left(\mu^2 - \left(m - \frac{y_0}{2\pi\alpha'}\right)^2\right)\frac{y_0}{2\pi\alpha'} \tag{2.27}$$

and we can easily check that in the relevant regime $\mu > m > \frac{y_0}{2\pi\alpha'} > 0$ the derivative of this expression with respect to y_0 is strictly positive. The D7-brane with strings attached can

continuously lower its energy until it turns into the black hole embedding with all strings dissolved in the D7-brane.

While none of the finite-density Minkowski embeddings solve the equations of motion, the trivial Minkowski embeddings with constant $A_t(r)$ do. As only F_{rt} enters the action and not $A_t(r)$ itself, these embeddings have $F = \Omega = d = 0$, and μ is a free parameter. For $\mu < m$ these are the only allowed configurations, so they dominate the ensemble. The fact that they are indistinguishable from the vacuum is completely natural from the field theory point of view: the lightest charge carriers have mass m , and for $\mu < m$ no non-zero density can be produced. When $\mu > m$ these solutions still exist, but in that regime the black hole embedding, with negative Ω , dominates the ensemble. At $\mu = m$ we hence have a second order phase transition between the trivial Minkowski embedding and the black hole embeddings. Both Ω and its first derivative with respect to μ are zero at that point, but the second derivative jumps from zero for the trivial Minkowski embedding to $-2\gamma\mathcal{N}(2\pi\alpha')^4 m^2$ for the black hole embedding. Again, this is exactly what we expect for a bosonic condensate with a non-trivial effective quartic interaction. The critical exponents are given by their mean-field values, that is, for $\mu = m + \epsilon$ we have $\Omega \sim \epsilon^2$, $c \sim \epsilon$, $d \sim \epsilon$.

In the canonical ensemble, the trivial Minkowski embeddings do not even contribute because they have $d = 0$. The finite-density Minkowski embeddings with strings attached that we considered above are not solutions of the equations of motion, so they cannot contribute. We thus conclude that, in the canonical ensemble, only black hole embeddings contribute. This analytically confirms the results of ref. [7], at least at zero temperature.

Lastly, notice that the configuration we mentioned above of a single straight string stretching between a D3-brane and a D7-brane in flat space is BPS, preserving half of the supersymmetry. We should ask why a density of such strings is not also BPS, and hence the dominant configuration? From a field theory point of view, the single string represents a single fundamental-representation “quark.” As the force between colored charges is Coulombic, we may introduce a single quark into the system and still make the total state of the system gauge invariant by placing an anti-fundamental color charge at infinity. A finite density of such states, as would be represented by the configuration we studied above, is not a physical configuration, however, because it has a finite color density. It does not represent a gauge invariant state that contributes to the partition function. Both at strong and weak coupling a scalar condensate forms at finite density, breaking the $SU(N_c)$ gauge group to $SU(N_c - 1) \times U(1)$. In other words, the system moves out onto the Higgs branch. No BPS states exist on the Higgs branch. For example, a finite density of baryons is gauge invariant but is not BPS. Our analysis is consistent with this field theory picture: the D7-brane-with-strings configuration is not a supergravity solution, and in the SYM theory we indeed find that a scalar condensate forms.

2.5 General zero-temperature probe D-brane embeddings

While the discussion so far was only for the D3/D7 system and the only field strength we turned on was F_{rt} , corresponding to finite density in the field theory, the same methods can be used to generate analytic embeddings for a variety of probe D-brane systems and worldvolume fields. In this subsection we will consider Dq-brane probes in the D3-brane

background, with $q = 5, 7$. The probe D5-brane wraps $AdS_4 \times S^2$ inside $AdS_5 \times S^5$, and corresponds to adding flavor fields confined to a (2+1)-dimensional defect in the SYM theory [19, 20].

We include F_{rt} and all possible gauge field components F_{ri} with spatial indices i parallel to the D3-branes, corresponding to non-vanishing $U(1)_B$ current components $\langle J_i \rangle$ in the field theory. We are working at zero temperature, so these currents will not dissipate and can be turned on as constant background fields. They do not require an electric field to be turned on. For all these cases, the warp factors continue to cancel in the DBI action, and we can obtain analytic solutions for the embeddings that basically correspond to D q -brane embeddings in flat space. For the interesting case of non-vanishing electric or magnetic fields, the warp factors do not cancel and one has to solve the full-fledged non-linear problem in curved space.

Our general D q -brane will be extended along $AdS_{D+2} \times S^n$, so that i runs from one to D . Generalizing eq. (2.4), the action for the D q -brane is

$$S_{Dq} = -N_f T_{Dq} \int d^{q+1} \xi \sqrt{-\det(g_{ab} + (2\pi\alpha') F_{ab})} \quad (2.28)$$

where T_{Dq} is the tension of the D q -brane. Introducing $y(r)$, $A_t(r)$ and $F_{ri} = A'_i(r)$, the Lagrangian for the D q -brane is

$$L = -\mathcal{N}_q r^n \sqrt{1 + y'^2 - (2\pi\alpha')^2 A_t'^2 + (2\pi\alpha')^2 \sum_i A_i'^2} \quad (2.29)$$

where $\mathcal{N}_q = N_f T_{Dq} V_n$ with V_n the volume of a unit n -sphere. We will divide both sides of eq. (2.28) by the volume of $\mathbb{R}^{D,1}$ so that $S_{Dq} = \int dr L$. To simplify the formulas let us introduce the following notation:

$$\vec{Y} = (y, (2\pi\alpha')A_t, (2\pi\alpha')A_i), \quad \vec{C} = \left(c, \frac{d}{(2\pi\alpha')}, \frac{J_i}{(2\pi\alpha')} \right) \quad (2.30)$$

that is, we construct a vector of fields and the associated conserved charges, with components Y_a and C_a . The inner product of these vectors has a Lorentzian signature, i.e. $diag(+1, -1, +1, +1, +1)$ for the D7-brane. With this, we have

$$L = -\mathcal{N}_q r^n \sqrt{1 + Y'^2} \quad (2.31)$$

and from this we get

$$C_a = \frac{\mathcal{N}_q r^n Y'_a}{\sqrt{1 + Y'^2}} \quad (2.32)$$

We algebraically solve for the fields, and find the regulated on-shell action,

$$Y_a'^2 = \frac{C_a^2}{\mathcal{N}_q^2 r^{2n} - C^2}, \quad S_{Dq} = -\mathcal{N}_q \int_0^\Lambda dr r^n \sqrt{\frac{\mathcal{N}_q^2 r^{2n}}{\mathcal{N}_q^2 r^{2n} - C^2}}. \quad (2.33)$$

Once again, the lower endpoint of integration depends on the sign of C^2 . The divergence in S_{Dq} is cancelled by the action for a single straight D q -brane with zero field strength,

$$S_0 = -\mathcal{N}_q \int_0^\Lambda dr r^n = -\frac{\mathcal{N}_q}{n+1} \Lambda^{n+1}, \quad S_{\text{ren}} = \lim_{\Lambda \rightarrow \infty} (S_{Dq} - S_0). \quad (2.34)$$

$\frac{d^2}{(2\pi\alpha')^2} - c^2 < 0$ for the D7-brane becomes $C^2 > 0$ for the Dq-brane. The Dq-brane has a turn-around point at $r_0 = \left(\frac{C^2}{\mathcal{N}_q^2}\right)^{\frac{1}{2n}}$. The integrals can be done using the machinery in the appendix by changing coordinates to $t = r_0^{2n}/r^{2n}$,

$$Y_a = C_a \frac{1}{2n} \mathcal{N}_q^{-\frac{1}{n}} (C^2)^{\frac{1-n}{2n}} \left(B\left(\frac{n-1}{2n}, \frac{1}{2}\right) - B\left(\frac{r_0^{2n}}{r^{2n}}; \frac{n-1}{2n}, \frac{1}{2}\right) \right), \quad (2.35)$$

$$S_{\text{ren}} = -\frac{1}{2n} \mathcal{N}_q^{-\frac{1}{n}} (C^2)^{\frac{1+n}{2n}} B\left(-\frac{n+1}{2n}, \frac{1}{2}\right) \quad (2.36)$$

Notice $S_{\text{ren}} < 0$, so the free energy is positive and hence these embeddings are thermodynamically disfavored relative to the embedding with a straight brane/anti-brane pair (which still has $\Omega = 0$), just as in section 2.2.

$\frac{d^2}{(2\pi\alpha')^2} - c^2 > 0$ for the D7-brane becomes $C^2 < 0$ for the Dq-brane. The Dq-brane does not have a turn-around point and the integrations extend to $r = 0$. Here the change of variables is to $u = \frac{\mathcal{N}_q^2}{|C^2|} r^{2n}$, with the result

$$Y_a = C_a \frac{1}{2n} \mathcal{N}_q^{-\frac{1}{n}} |C^2|^{\frac{1-n}{2n}} B\left(\frac{\mathcal{N}_q^2 r^{2n}}{\mathcal{N}_q^2 r^{2n} + |C^2|}; \frac{1}{2n}, \frac{n-1}{2n}\right) \quad (2.37)$$

$$S_{\text{ren}} = -\frac{1}{2n} \mathcal{N}_q^{-\frac{1}{n}} |C^2|^{\frac{1+n}{2n}} B\left(\frac{2n+1}{2n}, -\frac{n+1}{2n}\right) \quad (2.38)$$

As in the $\frac{d^2}{(2\pi\alpha')^2} - c^2 > 0$ case for the D7-brane, $S_{\text{ren}} > 0$, so the free energy is negative and these embeddings are thermodynamically favored.

3. The limit of zero mass

In this section we study the SYM theory at finite temperature and density, and take $m = 0$. At finite temperature, the AdS_5 metric in eq. (2.2) becomes the AdS-Schwarzschild metric,

$$ds^2 = \frac{dr_6^2}{f(r_6)} - f(r_6) dt^2 + r_6^2 d\vec{x}^2 + d\theta^2 + \sin^2 \theta ds_{S^1}^2 + \cos^2 \theta ds_{S^3}^2 \quad (3.1)$$

where $f(r_6) = r_6^2 - \frac{r_H^4}{r_6^2}$. The black hole horizon is related to the temperature by $r_H = \pi T$. The D7-brane embedding is described by $\theta(r_6)$. Introducing the worldvolume gauge field $F_{r_6 t} = \partial_{r_6} A_t(r_6)$, the D7-brane action eq. (2.4) becomes

$$S_{D7} = -\mathcal{N} \int^\Lambda dr_6 r_6^3 \cos^3 \theta \sqrt{1 + f(r_6) \theta'^2 - (2\pi\alpha')^2 A_t'^2} \quad (3.2)$$

where we have divided by the volume of $\mathbb{R}^{3,1}$, \mathcal{N} is the same as defined above, and primes denote $\frac{\partial}{\partial r_6}$. At this point we can plainly see that the system has only one conserved charge,

$$d \equiv \mathcal{N} r_6^3 \cos^3 \theta \frac{(2\pi\alpha')^2 A_t'}{\sqrt{1 + f(r_6) \theta'^2 - (2\pi\alpha')^2 A_t'^2}} \quad (3.3)$$

The algebraic solution for $A'_t(r_6)$ is

$$A'_t(r_6) = \frac{d}{(2\pi\alpha')^2} \sqrt{\frac{1 + f(r_6)\theta'^2}{\mathcal{N}^2 r_6^6 \cos^6 \theta + \frac{d^2}{(2\pi\alpha')^2}}}. \quad (3.4)$$

Inserting this into S_{D7} , we find the on-shell action,

$$S_{D7} = -\mathcal{N}^2 \int^\Lambda dr_6 r_6^6 \cos^6 \theta \sqrt{\frac{1 + f(r_6)\theta'^2}{\mathcal{N}^2 r_6^6 \cos^6 \theta + \frac{d^2}{(2\pi\alpha')^2}}}. \quad (3.5)$$

We may obtain the equation of motion for $\theta(r_6)$ in two ways. We may either derive it from eq. (3.2) and then plug in the gauge field eq. (3.4), or we may eliminate the gauge field at the level of the action via a Legendre transform and then derive $\theta(r_6)$'s equation of motion. The Legendre-transformed action, \bar{S}_{D7} , is

$$\bar{S}_{D7} = S_{D7} - \int dr_6 F_{r_6 t} \frac{\delta S_{D7}}{\delta F_{r_6 t}} = - \int dr_6 \sqrt{1 + f(r_6)\theta'^2} \sqrt{\mathcal{N}^2 r_6^6 \cos^6 \theta + \frac{d^2}{(2\pi\alpha')^2}}. \quad (3.6)$$

Once again, to cancel the $\Lambda \rightarrow \infty$ divergence in S_{D7} we subtract the action of a single straight D7-brane with no gauge field. At finite temperature, the lower endpoint of integration is now r_H , hence

$$S_0 = -\mathcal{N} \int_{r_H}^\Lambda dr_6 r_6^3 = -\frac{1}{4} \mathcal{N} \Lambda^4 + \frac{1}{4} \mathcal{N} r_H^4, \quad (3.7)$$

and S_{ren} is defined as above.

A valid solution of $\theta(r_6)$'s equation of motion is simply $\theta(r_6) = 0$, corresponding to $m = 0$ and $\langle \mathcal{O}_m \rangle = 0$ in the SYM theory. For this solution, the integrals for $A_t(r)$ and S_{D7} are identical to those for a black hole embedding with $c = 0$, eqs. (2.9) and (2.10), but with $r \rightarrow r_6$ and with the lower endpoint of integration r_H . We thus find

$$A_t(r_6) = \frac{1}{6} \mathcal{N}^{-1/3} (2\pi\alpha')^{-4/3} d^{1/3} \left(B \left(\frac{\mathcal{N}^2 r_6^6}{\mathcal{N}^2 r_6^6 + \frac{d^2}{(2\pi\alpha')^2}}; \frac{1}{6}, \frac{1}{3} \right) - B \left(\frac{\mathcal{N}^2 r_H^6}{\mathcal{N}^2 r_H^6 + \frac{d^2}{(2\pi\alpha')^2}}; \frac{1}{6}, \frac{1}{3} \right) \right) \quad (3.8)$$

Notice that $A_t(r_H) = 0$. The value of μ is given by taking $r_6 \rightarrow \infty$, for which the first term in parentheses goes to $B(\frac{1}{6}, \frac{1}{3})$. For the on-shell action we find

$$S_{\text{ren}} = -\frac{1}{4} \mathcal{N} r_H^4 - \frac{1}{6} \mathcal{N}^{-1/3} \left(\frac{d}{2\pi\alpha'} \right)^{4/3} B \left(\frac{\frac{d^2}{(2\pi\alpha')^2}}{\mathcal{N}^2 r_H^6 + \frac{d^2}{(2\pi\alpha')^2}}; -\frac{2}{3}, \frac{7}{6} \right). \quad (3.9)$$

The $T = 0$ limit of course reproduces the results of section 2.3 for black hole embeddings with $c = 0$. A straightforward analysis reveals that $\Omega = -S_{\text{ren}}$ has no meaningful features: in the $m = 0$ limit, the system is stable and exhibits no phase transitions.

4. Conclusion

For N_f massive $\mathcal{N} = 2$ supersymmetric hypermultiplet fields at finite $U(1)_B$ density coupled to the $\mathcal{N} = 4$ SYM theory in the limits of large- N_c and large λ , we have provided analytic expressions for the hypermultiplets' leading-order (in N_f/N_c) contribution to the free energy in two limits: zero temperature and zero mass. We have found a second order transition at zero temperature when $\mu = m$. In the supergravity description, we have confirmed the proposal of refs. [7, 18] that, in the canonical ensemble, only black hole D7-brane embeddings are physical, while in the grand canonical ensemble black hole embeddings are appropriate for $\mu > m$ while Minkowski embeddings with constant gauge field (and zero field strength) are appropriate when $\mu < m$. Perhaps the most obvious task for the future is a thorough analysis of the phase diagram of this theory in the grand canonical ensemble.

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A. Beta functions and incomplete Beta functions

The integrals needed in the bulk of the text all take the form of Beta functions or incomplete Beta functions. For the reader's convenience we collect some of the important properties of these functions in this appendix. The Beta function is defined as

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dt(1-t)^{b-1}t^{a-1} = \int_0^\infty du(1+u)^{-(a+b)}u^{a-1} \quad (\text{A.1})$$

and the incomplete Beta function as

$$B(x; a, b) = \int_0^x dt(1-t)^{b-1}t^{a-1} = \int_0^{x/(1-x)} du(1+u)^{-(a+b)}u^{a-1}. \quad (\text{A.2})$$

These satisfy the recursion relation

$$B(x; a, b) = B(a, b) - B(1-x; b, a) \quad (\text{A.3})$$

and are related to the hypergeometric function as

$$B(x; a, b) = a^{-1}x^a F(a, 1-b; a+1; x). \quad (\text{A.4})$$

The Λ^4 divergence in S_{D7} appears in the expansion, for $a = -2/3$ about $x = 0$,

$$B\left(x; -\frac{2}{3}, b\right) = -\frac{3}{2}x^{-2/3} + O(x^{1/3}) \quad (\text{A.5})$$

where for S_{D7} we will have $x \sim \Lambda^{-6}$ (see for example eq. (A.8) below).

The integrals we need can easily be brought into the form of eq. (A.2). For notational simplicity, we will replace $\frac{d^2}{(2\pi\alpha')^2}$ with just d^2 . We begin with the $d^2 - c^2 < 0$ case of section 2.2. The integral for $y(r)$ and $A_t(r)$, eq. (2.9), in this case is

$$I_{<}(r) = \int_{r_0}^r dr (\mathcal{N}^2 r^6 - (c^2 - d^2))^{-1/2}, \quad (\text{A.6})$$

with $r_0 = \mathcal{N}^{-1/3} (c^2 - d^2)^{1/6}$. We change variables to $t = r_0^6/r^6$,

$$\begin{aligned} I_{<}(r) &= \frac{1}{6} \mathcal{N}^{-1/3} (c^2 - d^2)^{-1/3} \int_t^1 dt (1-t)^{-1/2} t^{-2/3} \\ &= \frac{1}{6} \mathcal{N}^{-1/3} (c^2 - d^2)^{-1/3} \left(B\left(\frac{1}{3}, \frac{1}{2}\right) - B\left(\frac{r_0^6}{r^6}, \frac{1}{3}, \frac{1}{2}\right) \right). \end{aligned} \quad (\text{A.7})$$

The same change of variables works for the regulated on-shell action eq. (2.10),

$$\begin{aligned} S_{D7}^{\leq} &= -\frac{1}{6} \mathcal{N} r_0^4 \int_{r_0^6/\Lambda^6}^1 dt (1-t)^{-1/2} t^{-5/3} \\ &= -\frac{1}{6} \mathcal{N} r_0^4 \left(B\left(-\frac{2}{3}, \frac{1}{2}\right) - B\left(\frac{r_0^6}{\Lambda^6}, -\frac{2}{3}, \frac{1}{2}\right) \right) \\ &= -\frac{1}{6} \mathcal{N} r_0^4 B\left(-\frac{2}{3}, \frac{1}{2}\right) - \frac{1}{4} \mathcal{N} \Lambda^4 + O(\Lambda^{-2}) \end{aligned} \quad (\text{A.8})$$

Now we turn to the $d^2 - c^2 > 0$ case of section 2.3. The integral for $y(r)$ and $A_t(r)$ is

$$I_{>}(r) = \int_0^r dr (\mathcal{N}^2 r^6 + d^2 - c^2)^{-1/2}. \quad (\text{A.9})$$

We change variables to $u = \frac{\mathcal{N}^2 r^6}{d^2 - c^2}$,

$$\begin{aligned} I_{>}(r) &= \frac{1}{6} \mathcal{N}^{-1/3} (d^2 - c^2)^{-1/3} \int_0^u du (1+u)^{-1/2} u^{-5/6} \\ &= \frac{1}{6} \mathcal{N}^{-1/3} (d^2 - c^2)^{-1/3} B\left(\frac{\mathcal{N}^2 r^6}{\mathcal{N}^2 r^6 + d^2 - c^2}, \frac{1}{6}, \frac{1}{3}\right). \end{aligned} \quad (\text{A.10})$$

The regulated on-shell action is, with $\Lambda' = \frac{\mathcal{N}^2 \Lambda^6}{d^2 - c^2}$

$$\begin{aligned} S_{D7}^{>} &= -\frac{1}{6} \mathcal{N}^{-1/3} (d^2 - c^2)^{2/3} \int_0^{\Lambda'} du (1+u)^{-1/2} u^{1/6} \\ &= -\frac{1}{6} \mathcal{N}^{-1/3} (d^2 - c^2)^{2/3} \left(B\left(\frac{7}{6}, -\frac{2}{3}\right) - B\left(\frac{1}{\Lambda' + 1}, -\frac{2}{3}, \frac{7}{6}\right) \right) \\ &= -\frac{1}{6} \mathcal{N}^{-1/3} (d^2 - c^2)^{2/3} B\left(\frac{7}{6}, -\frac{2}{3}\right) - \frac{1}{4} \mathcal{N} \Lambda^4 + O(\Lambda^{-2}) \end{aligned} \quad (\text{A.11})$$

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